Exercise 49

Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a.

$$\text{(a)}\quad f(x) = \frac{x^4 - 1}{x - 1}, \quad a = 1$$

$$\text{(b)}\quad f(x) = \frac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$$

$$\text{(c)}\quad f(x) = [\![\sin x]\!], \quad a = \pi$$

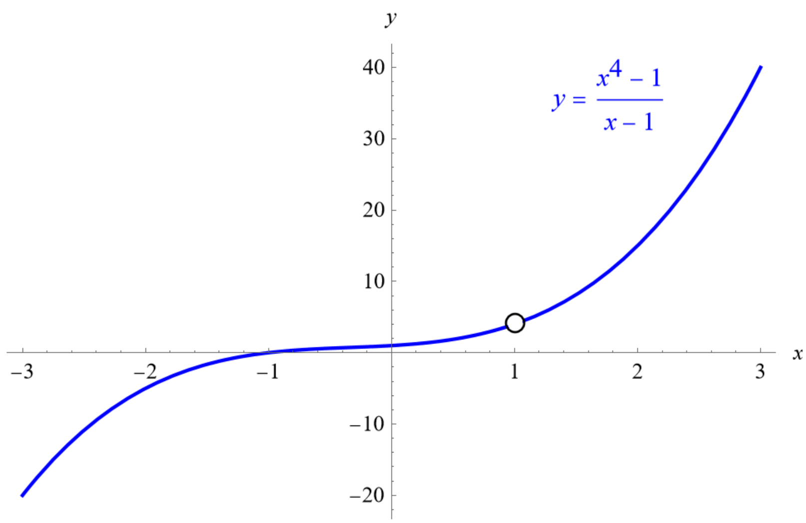
Solution

Part (a)

$$f(x) = \frac{x^4 - 1}{x - 1}, \quad a = 1$$

The factor of $x - 1$ in the denominator would cause a vertical asymptote to appear in the graph at $x = 1$, but since one of the factors in the numerator cancels out with it, there's only a hole at $x = 1$.

$$f(x) = \frac{(x^2+1)(x^2-1)}{x-1} = \frac{(x^2+1)(x+1)(x-1)}{x-1} = (x^2+1)(x+1)$$



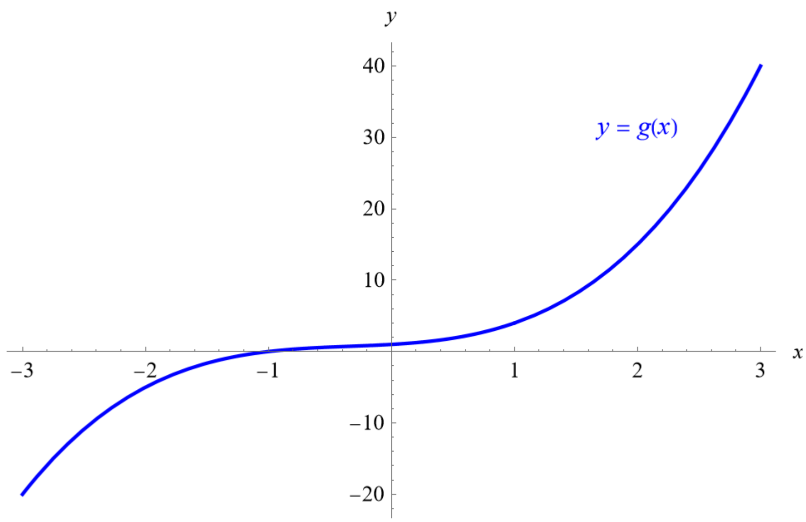
To remove the hole, figure out what $f(x)$ would be at $x = 1$.

$$(1^2 + 1)(1 + 1) = (2)(2) = 4$$

Then define a new function with this value at $x = 1$.

$$g(x) = \begin{cases} \dfrac{x^4 - 1}{x - 1} & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$$

The graph of $g(x)$ versus x is continuous.

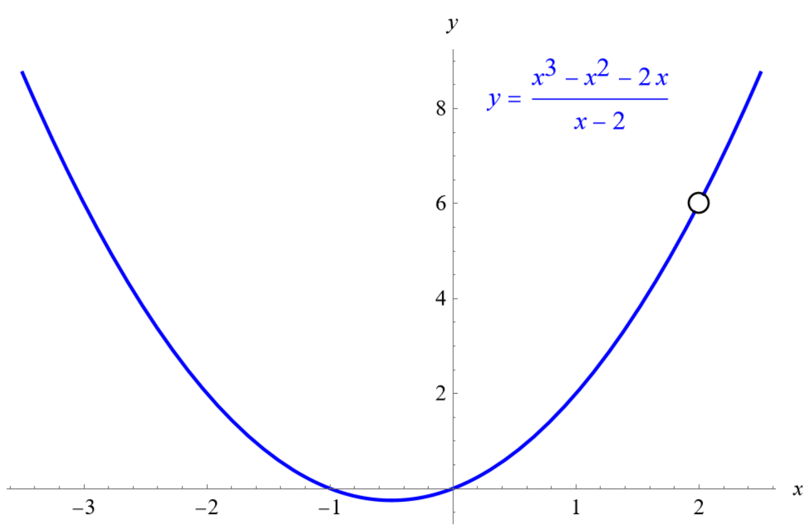


Part (b)

$$f(x) = \frac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$$

The factor of $x - 2$ in the denominator would cause a vertical asymptote to appear in the graph at $x = 2$, but since one of the factors in the numerator cancels out with it, there's only a hole at $x = 2$.

$$f(x) = \frac{x(x^2 - x - 2)}{x - 2} = \frac{x(x + 1)(x - 2)}{x - 2} = x(x + 1)$$



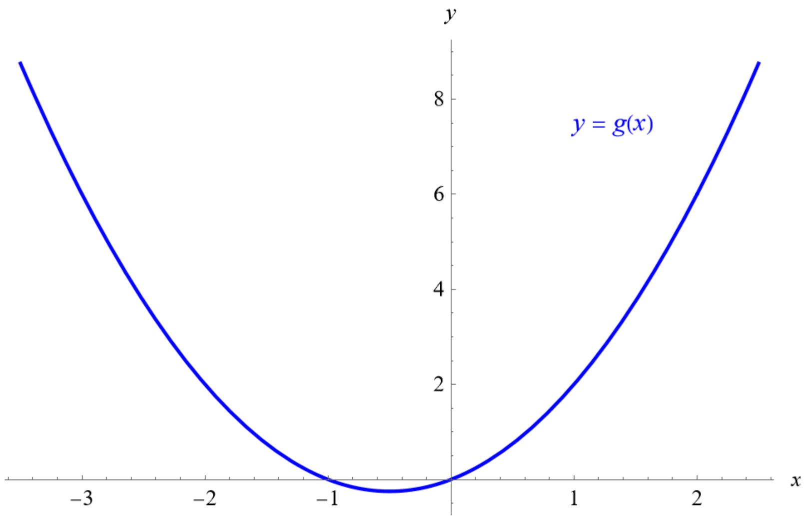
To remove the hole, figure out what $f(x)$ would be at $x = 2$.

$$2(2 + 1) = (2)(3) = 6$$

Then define a new function with this value at $x = 2$.

$$g(x) = \begin{cases} \dfrac{x^3 - x^2 - 2x}{x - 2} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$$

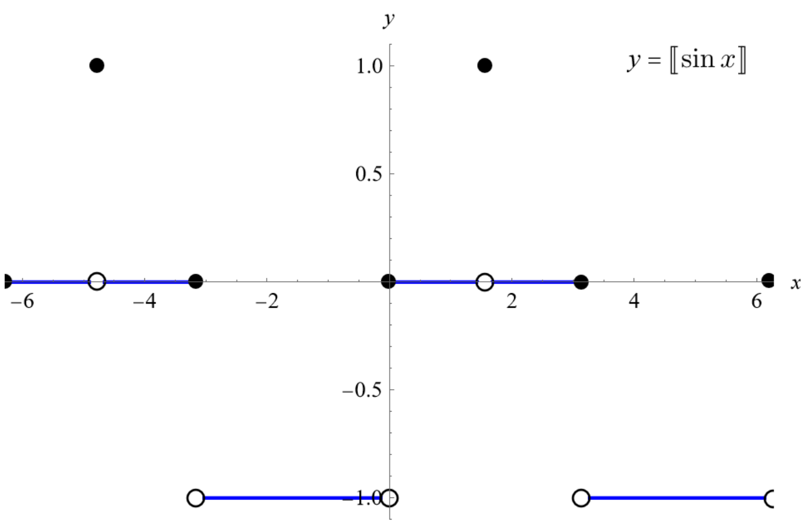
The graph of $g(x)$ versus x is continuous.



Part (c)

$$f(x) = [\![\sin x]\!], \quad a = \pi$$

Below is a graph of this function versus x.



As the graph shows, there are removable discontinuities at $x = \pi/2 + 2n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. At $x = \pi$, however, there's a jump discontinuity.